A branching particle system as a model of FKPP fronts.

1) Motivation from population dynamics / PDE theory.

population invading a one-dimensional habitat

FKPP reaction-diffusion equation

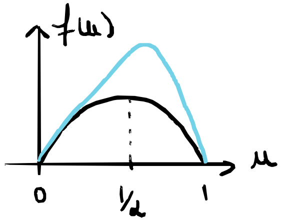
$$\partial_t u = \underbrace{\tfrac{1}{2}\partial_{xx} u}_{\text{migrations}} + \overbrace{f(u)}^{\text{local birth/death dyn.}}$$

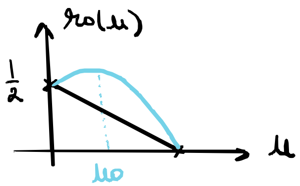
$u(t,x)$ density of individuals at x at time $t>0$

$u \in [0,1]$.

EXAMPLES

(a) $f(u) = \frac{1}{2}\,\overbrace{u(1-u)}^{\text{saturation}}$





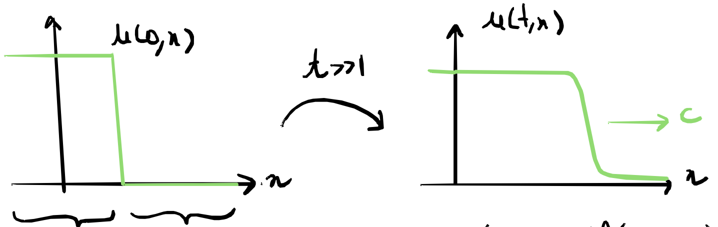
$r_0(u) = \frac{f(u)}{u}$ per capita growth rate

(b) $f(u) = \frac{1}{2}u(1-u)\overbrace{(1+Bu)}^{\text{cooperation}}$ $\quad B>0$

if $B\leq 1$, r_0 max at $u=0$

if $B>1$, r_0 max at $u_0>0$ cooperation Allee effect.

Q1: Macroscopic dynamic?



colonised unoccupied

$u(t,x) \approx \varphi(x-ct)$

$u(0,x) = \mathbb{1}_{x<x_0}$

$(t,x) \longmapsto \varphi(x-ct)$ Travelling wave solution

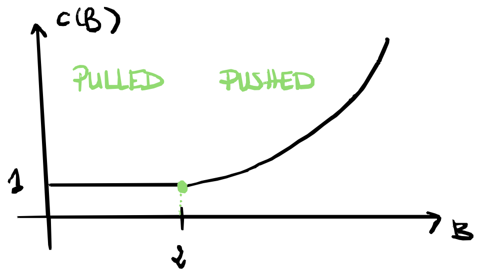
constant profile speed of invasion

two types of waves

1) if $c=\sqrt{2f'(0)}$, the wave is pulled

2) if $c>\sqrt{2f'(0)}$, the wave is pushed.

Ex: $f(u) = \frac{1}{2}u(1-u)(1+Bu)$ $\qquad f'(0)=\frac{1}{2}$



Q2 : Fluctuations ?

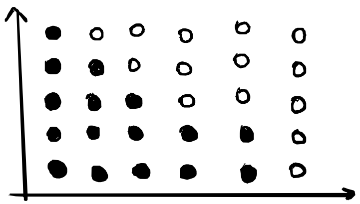
FKPP equation = hydrodynamic limit of some IBM
When taking this limit, what is the order of the fluctuations around the deterministic limit ?

Simulations by Birzu, Hallatschek and Korolev 2018.

Wright-Fisher model with vacancies.

* discrete space/time
* on each deme : N sites that can be ● occupied or O vacant
* at each time step

(1) migration (2) duplication



X_t : position of the front (e.g. rightmost particle)

for $N \gg 1$ $X_t \approx ct$ and $\langle (X_t - ct)^2 \rangle \approx \sigma(N)^2 t$

$B \in (0,2)$	$B \in (2,4)$	$B > 4$
$\log(N)^{-3}$	$N^{1-\alpha}$	N^{-1} (CLT)

$$\alpha = \frac{c^2 + \sqrt{c^2-1}}{c^2 - \sqrt{c^2-1}} \in (1,2)$$

PULLED SEMIPUSHED FULLY PUSHED

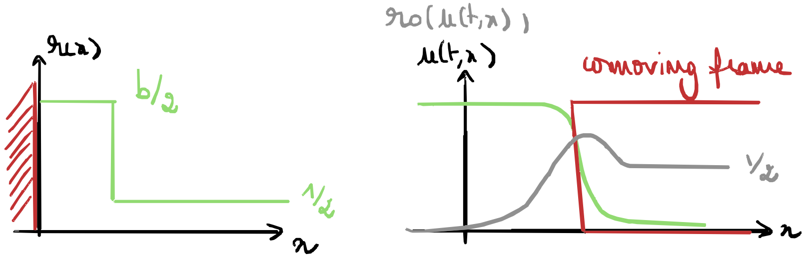
2) A toy model

Dyadic Branching Brownian motion (BBM) on $[0,+\infty)$

(i) inhomogeneous branching rate $r(x)$

(ii) critical drift $-\mu$.

(iii) killing at 0.



$b > 1$

INTERPRETATION

comoving frame $u \approx 0$ ⇒ BBM (linearisation)
(independence + exponential growth)

(i) r approximation of r_0 in the comoving frame

b : strength of cooperation in the BBM.

(ii) μ speed of invasion (= speed of the frame)

(iii) particles from the bulk do not contribute to the invasion ($r_0 \approx 0$)

How to choose μ? spectral theory.

Consider the truncated system killed at 0 and $L \gg 1$

$\mathcal{N}_t^L$: set of particles that stayed in $[0,L]$ until t.

many-to-one lemma $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ measurable

$$E_x\Big[\sum_{v \in \mathcal{N}_t^L} f(\underbrace{X_v(t)}_{\text{position of } v})\Big] = \int_{y=0}^{L} f(y)\, p_t(x,y)\, dy$$

where p_t is the fundamental solution of the linear PDE

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_{xx} u + \mu \partial_x u + r(x) u \\ u(t,L) = u(t,0) = 0 \end{cases}$$

Sturm-Liouville theory

eigenvalues/vectors self-adjoint SL problem

$$p_t(x,y) = \underbrace{e^{\mu(x-y)}}_{\text{"makes the operator self adjoint"}} e^{\frac{1-\mu^2}{2}t} \sum_{i=1}^{+\infty} e^{\lambda_i t} v_i(x) v_i(y)$$

with $\lambda_1 > \lambda_2 > \dots > \lambda_m > \dots \longrightarrow -\infty$

$\lambda_1 \xrightarrow{L\infty} \lambda_1^\infty$ generalized principal eigenvalue

$\lambda_1^\infty - \lambda_1 \sim e^{-2\beta L}$

For $t \gg 1$, the sum should be dominated by its first term

$$p_t(x,y) \approx e^{(\lambda_1 + \frac{1-\mu^2}{2})t}\, e^{\mu x} v_1(x)\, e^{-\mu y} v_1(y)$$

we choose $\mu = \sqrt{1+2\lambda_1^\infty}$. (neither ↗ nor ↘ exponentially fast)

$$\approx e^{(\lambda_1 - \lambda_1^\infty)t}\, h(x)\, \tilde{h}(y)$$

mass loss ↗ stable configuration

3) The pulled/pushed regime in the BBM

There exists $b_1 > 1$ such that

(i) for all $b \in [1, b_1]$, $\lambda_1^\infty = 0$ and $\mu = 1$ PULLED

(ii) for all $b > b_1$, $\lambda_1^\infty \nearrow b$ (so does μ) PUSHED

↳ same shift in the speed of invasion as in the SPDE

Define $\beta = \sqrt{2\lambda_1^\infty}$.

For $b > b_1$, $v_1(y) \propto e^{-\beta y}$

$\tilde{h}(y) \propto e^{-(\mu+\beta)y}$

Dispersion relation in pushed fronts.

$$\varphi(z) \propto e^{-(c+\sqrt{c^2-1})z}$$

↳ Same relation between the decay and the speed of the front in the PDE and in the BBM.

In expectation, the BBM has the same macroscopic behaviour as the FKPP front.

4) The semi/fully pushed regimes in the BBM

Fluctuations in the position of the fronts

↳ Fluctuations in the population size in the comoving frame

Z_t^L : # particles in the BBM at time t. (in the truncated BBM)

At $t=0$, N particles distributed according to $\tilde h$.

↑ demographic parameter.

Theorem 1 (T. 24) The semipushed regime

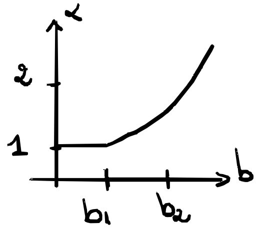
There exists $b_2 > b_1$ such that, for all $b \in (b_1, b_2)$

* $\alpha(b) = \frac{\mu+\beta}{\mu+\beta} \in (1,2)$
* $\left(\frac{1}{N} Z_{N^{\alpha-1}t}\right)$ converges in distribution to an α-stable CSBP

Theorem 2 (Schertzer-T. 24) The fully pushed regime

For $b > b_2$, $\alpha(b) > 2$ and

$\left(\frac{1}{N} Z_{tN}\right)$ converges in distribution to a Feller diffusion



↳ Same time scales

5) Heuristics.

1ST moment

$\langle \tilde h, 1\rangle = 1$

$$\mathbb{E}_x[Z_t^L] = \int_{y=0}^{L} p_t(x,y) \approx e^{(\lambda_1-\lambda_1^\infty)t}\, \underbrace{h(x)}_{\text{contribution of a particle at } x}$$

$$\mathbb{E}[Z_t] \approx N\int h(x)\tilde h(x)\,dx = N$$

$t \ll (\lambda_1^\infty-\lambda_1)^{-1}$ $\qquad \langle h, \tilde h\rangle = 1$

$\Rightarrow$ critical CSBP.

2ND moment

many-to-two lemma

$$\mathbb{E}_x[Z_t^{L(2)}] = \int_{s=0}^{t}\int_{y=0}^{L} \overbrace{p_s(x,y)}^{\approx h(x)\tilde h(y)} \cdot 2r(y)\, \underbrace{\mathbb{E}_y[Z_{t-s}^L]}_{\approx h(y)}{}^2\, dy\, ds$$

$$\approx t\,h(x)\int_0^L h(y)^2\,\tilde h(y)\,dy.$$

$h^2(y)\tilde h(y) \propto e^{(\mu-3\beta)y}$ integrable iff $\boxed{\alpha > 2}$

(i) if $\alpha > 2$, $L \to +\infty$ finite moments : CLT, Feller diffusion

(ii) if $\alpha < 2$, infinite moments

introduction of a cut-off L to identify the main contributions

$h(L) = N \iff L = \frac{1}{\mu-\beta}\log(N)$

For this choice of L: $(\lambda_1^\infty-\lambda_1)^{-1} = N^{\alpha-1}$

$$\mathbb{P}_L(Z_t > xN) \sim \frac{1}{x^\alpha}$$